STRICT FUNDAMENTAL DOMAINS FOR GROUP ACTIONS

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Abstract. For a finite group action $G \curvearrowright X$, the canonical quotient map does not have a section in general. This note shows that if G is an orthogonal subgroup and the section exists, then G is generated by reflections.

1. Introduction

Let G be a finite group acting on a space X. A **fundamental domain for the G-action on X** is a subset D of X such that the map

$$D \hookrightarrow X \twoheadrightarrow X/G$$

is a bijection. The fundamental domain D is called **strict** if the above map is a homeomorphism. Clearly, the existence of a section of the quotient map $X \twoheadrightarrow X/G$ is equivalent to that of a strict fundamental domain for the G-action on X.

A fundamental domain always exists while a strict one does not in general. This paper provides a result on the condition of G to have a strict fundamental domain.

Theorem 1.1. *Given $G \leq \mathsf{O}(\mathbb{R}^{n+1})$ nontrivial and finite, the following are equivalent:*

(a) *The G-action on $\mathbb{R}^{n+1}$ has a strict fundamental domian.*

(b) *The group G is a Coxeter group, that is, generated by reflections.*

The implication $(b) \Rightarrow (a)$ is a well-known result; see [Hum92, Theroem 1.12]. So we will only prove $(a) \Rightarrow (b)$ in Theorem 3.6 by induction on n and the slice theory. Explicitly, we focus on a neighborhood of a point $x \in S^n \subset \mathbb{R}^n$ with the stablizer G_x nontrivial, and this leads to the G_x-action on a sphere $S^{n-1} \subset T_x S^n$, where the dimension of the sphere goes down.

Since every linear representation of a finite group can be orthogonal, we obtain: A finite linear group has a strict fundamental domain if and only if the group is generated by reflections.

One motivation is the works of [Hor+24; Son22; Gon24; CGS24; GHL25]. They studied the Coxeter group action on the corresponding toric variety or its cohomology, for which the implication $(b) \Rightarrow (a)$ was applied. One consequence after this paper is that there is not a straightforward way to generalize the above results to other groups.

2. Preparation

In this section, we take $G \leq \mathbf{O}(\mathbb{R}^{n+1})$ as a finite subgroup unless specified otherwise. We will prepare for the main result: for the G-action, it is sufficient to focus on a sphere for a strict fundamental domain , and on a neighborhood of the sphere for the group action.

2.1. Focus on S^n.

The existence of a strict fundamental domain for the G-action on $\mathbb{R}^{n+1}$ is equivalent to that on S^n, which follows from the results in Lemmas 2.1 and 2.2 below.

Lemma 2.1. *Let X be a H-space with H a general finite group, D be a strict fundamental domian for the H-action on X. If Y is a H-subsapce of X, then $D \cap Y$ is a strict fundamental domain for for the H-action on Y.*

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Proof. It follows from that $D \cap Y \hookrightarrow Y \twoheadrightarrow Y/H$ is a bijection, which is clearly true. $\square$

Given topological subspace $X \subset \mathbb{R}^{n+1}$ and interval $I \subset \mathbb{R}$, we write $I \cdot X$ for the subsapce $\{tx \in \mathbb{R}^{n+1} : t \in I, x \in X\}$, and $\mathrm{cone}(X)$ for $[0, \infty) \cdot X$.

Lemma 2.2. *Suppose that $X \subset \mathbb{R}^{n+1}$ is G-invariant, $D \subset X$ is a strict fundamental domain for the G-action on X. Then $\mathrm{cone}(D)$ is a strict fundamental domain for the G-action on $\mathrm{cone}(X)$.*

Proof. Prove along the following steps:

- For $0 \le t_1 \le t_2$, the map $[t_1, t_2] \cdot D \hookrightarrow [t_1, t_2] \cdot X \twoheadrightarrow ([t_1, t_2] \cdot X)/G$ is a bijection.
- For each $i \in \mathbb{N}$, the homeomorphism $f_i : [i, i+1] \cdot D \to ([i, i+1]) \cdot X/G$ has an inverse g_i.
- $\cup g_i$ is inverse to the map $\mathrm{cone}(D) \to \mathrm{cone}(X)/G$, also continuous by the gluing lemma.

$\square$

2.2. **Local behaviors determine linear group action.** Suppose that $x \in \mathbb{R}^{n+1}$, and $\mathbb{R}^{n+1}$ has a basis $\{X_1, X_2, \dots, X_{n+1}\}$. Then there is an isomorphism of vector spaces

$$\mu_x : T_x\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \qquad \sum a_i \frac{\partial}{\partial X_i} \mapsto \sum a_i X_i,$$

which is actually independent of basis.

For $x \in \mathbb{R}^{n+1}$, there are identifications: the tangent space $T_x\mathbb{R}^{n+1}$ and the original Euclidean space $\mathbb{R}^{n+1}$ via μ_x, $\mathbb{R}^{n+1}$ with the affine space centered at x via the translation by x. Moreover, the two identifications are $\mathrm{GL}(\mathbb{R}^{n+1})$-equivariant, in the sense that the following diagram, for any $g \in \mathrm{GL}(\mathbb{R}^{n+1})$, is commutative.

$$\begin{tikzcd}
T_x\mathbb{R}^{n+1} \arrow[r, "\mu_x"] \arrow[d, "g_*"'] & \mathbb{R}^{n+1} \arrow[r, "+x"] \arrow[d, "g"'] & \mathbb{R}^{n+1} \arrow[d, "g"'] \\
T_{g(x)}\mathbb{R}^{n+1} \arrow[r, "\mu_{g(x)}"'] & \mathbb{R}^{n+1} \arrow[r, "+g(x)"'] & \mathbb{R}^{n+1}
\end{tikzcd}$$

Hence we can obtain:

Lemma 2.3. *If $g \in \mathrm{GL}(\mathbb{R}^{n+1})$, then g is determined by its behavior on a neighborhood in $\mathbb{R}^{n+1}$.*

For $x \in S^n$, define maps

$$\tau_x : T_xS^n \to \mathbb{R}^{n+1}, \qquad y \mapsto \mu_x(y) + x,$$

and

$$\eta : \mathbb{R}^{n+1} \setminus 0 \to S^n, \qquad y \mapsto \frac{y}{\langle y, y \rangle}.$$

We have an inclusion $\eta \circ \tau_x : T_xS^n \to S^n$ which identifies T_xS^n with an open subset of S^n centered at x. This identification is $\mathrm{O}(\mathbb{R}^{n+1})$-equivariant: $g \circ \eta \circ \tau_x = \eta \circ \tau_{g(x)} \circ g_*$ for $g \in \mathrm{O}(\mathbb{R}^{n+1})$, because the image of T_xS^n under μ_x is $(x)^\perp$ and the following diagrams are commutative:

$$\begin{tikzcd}
T_xS^n \arrow[r, hook] \arrow[d, "g_*"'] & T_x\mathbb{R}^{n+1} \arrow[r, "\mu_x"] \arrow[d, "g_*"'] & \mathbb{R}^{n+1} \arrow[r, "+x"] \arrow[d, "g"'] & \mathbb{R}^{n+1} \arrow[d, "g"'] \\
T_{g(x)}S^n \arrow[r, hook] & T_{g(x)}\mathbb{R}^{n+1} \arrow[r, "\mu_{g(x)}"'] & \mathbb{R}^{n+1} \arrow[r, "+g(x)"'] & \mathbb{R}^{n+1},
\end{tikzcd} \qquad \begin{tikzcd}
\mathbb{R}^{n+1} \setminus 0 \arrow[r, "\eta"] \arrow[d, "g"'] & S^n \arrow[d, "g"'] \\
\mathbb{R}^{n+1} \setminus 0 \arrow[r, "\eta"'] & S^n.
\end{tikzcd}$$

Hence we obtain:

Lemma 2.4. *If $g \in \mathrm{O}(\mathbb{R}^{n+1})$, then g is determined by its behavior on a neighborhood in S^n.*

Corollary 2.5. *If $H < G$ acts on an open set U (small enough) in S^n, then the H-action is effective.*

3. PROOF

In this section, we will prove the main result of this paper. Unless specified otherwise, $G \leq \mathbf{O}(\mathbb{R}^{n+1})$ is a nontrivial finite group, and $D \subset S^n$ is a strict fundamental doamin for the G-action on S^n. Then D is closed in S^n since S^n/G is compact.

For a point $x \in S^n$, it has a stablizer group $G_x \leq G$ that fixes x. We can choose a G_x-invariant open subset $U_x \subset S^n$ containing x, which satisfies that $gU_x \cap U_x = \emptyset$ for any $g \in G \setminus G_x$. This U_x is called a **slice at** x. Moreover, the slice U_x is called **good** if the closure $\overline{U_x} = \eta \circ \mu_x(B)$ for some bounded disk $B \subset T_x S^n$. Clearly, $G_y \leq G_x$ for any $y \in U_x$.

Lemma 3.1. *Every point $x \in S^n$ has a slice for the G-action, which can be good.*

Proof. Since S^n is Hausdorff and G is finite, we can choose an open neighborhood V_y for each point $y \in G(x)$ such that those open neighborhoods are disjoint. Then the open set

$$U_x := \bigcap_{y \in G(x)} \bigcap_{g(y)=x} gV_y$$

is a slice at x. Moreover, we can choose V_x to lie in a bounded subset of $\eta\tau_x\left(T_x S^n\right)$, then U_x is good since $\eta\tau_x$ is G_x-equivariant. □

Lemma 3.2. *A point $x \in D$ has slice $U_x \subset S^n$ small enough, the space $U_x \cap D$ is a strict fundamental domain for the G_x-action on U_x.*

Proof. Since D is closed, one can choose U_x small enough such that the following commutative diagram holds:

$$\begin{tikzcd}
U_x \cap D \arrow[r, hook] \arrow[d, equal] & U_x \arrow[r, two heads] \arrow[d, hook] & U_x/G_x \arrow[d, "\cong"] \\
\left(\bigcup_{g \in G} gU_x\right) \cap D \arrow[r, hook] & \bigcup_{g \in G} gU_x \arrow[r, two heads] & \left(\bigcup_{g \in G} gU_x\right) \Big/ G.
\end{tikzcd}$$

Then the top map is a homeomorphism. □

Remark 3.3. One can refer to [Bre72, Section II.4] for the above results when G is a general group.

Let $\mathrm{Int}(D)$ denote the interior of D, and ∂D the boundary. Then $D = \mathrm{Int}(D) \sqcup \partial D$.

Lemma 3.4. *Let $x \in D$. The stablizer $G_x = \langle e \rangle$ iff $x \in \mathrm{Int}(D)$.*

Proof. When $x \in \mathrm{Int}(D)$, we can choose the slice U_x at x to lie in D. If $G_y = G_x$ for any $y \in U_x$, then G_x acts trivially on U_x and hence has to be $\langle e \rangle$ by Corollary 2.5. If there is a point $x' \in U_x$ with $G_{x'} \lneqq G_x$, then there is some $g \in G_x \setminus G_{x'}$ such that $x' \neq g(x') \in U_x$, which contradicts against the definition of D. Therefore, $G_x = \langle e \rangle$.

When $G_x = \langle e \rangle$, we have $g(x) \notin D$ for all $g \neq e$. We can choose U_x small enough such that $g(U_x) \cap D = \emptyset$ for all $g \neq e$. By definition of D, we have $U_x \cap D = U_x$ and $x \in \mathrm{Int}(D)$. □

We can get the following results directly from Lemma 3.4 and its proof.

Corollary 3.5.

(i) *There exists a point in S^n with a trivial stablizer.*
(ii) *The subspace $S^n \setminus \cup_{g \neq e}(S^n)^g$ is homeomorphic equivariantly to $G \times \mathrm{Int}(D)$.*
(iii) *For any $g \in G$, the map $g : D \to gD$ perserves interior points, as well as boundary points.*
(iv) *For any $g \in G$, the intersection $gD \cap D$ lies in ∂D.*
(v) *The condition "small enough" in Lemma 3.2 is unnecessay.*

Now we can prove the main result of this paper.

Theorem 3.6. *The group G is generated by reflections.*

Proof. We use induction. When $n = 0$, the group $\mathrm{O}(\mathbb{R}^1) = \mathbb{Z}/2$ is generated by a reflection. Then it is clear that the action $G \curvearrowright S^0$ (or equivariantly, $G \curvearrowright \mathbb{R}^1$) has a strict fundamental domain iff G is generated by the unique reflection.

For a greater n, we study the stablizer G_x at a point $x \in \partial D$, which is nontrivial by Lemma 3.4. Choose a good slice $U_x \subset S^n$ at x such that $\overline{U_x}$ is the image of a closed bounded disk B in $T_x S^n$ under the map $\eta\tau_x$. Then the G_x-action on ∂B has a strict fundamental domain due to Lemmas 2.1 and 3.2. The induction hypothesis says that G_x acts as reflections on $\partial B \cong S^{n-1}$, and hence on B, on U_x and on S^n.

We can choose finite such good slices $U_x \subset S^n$ as above to cover the closed subsapce $\bigcup_{g\in G} g(\partial D)$. We claim that the group G is generated by the corresponding stablizers G_x; that is,

$$G' := \left\langle G_x \,:\, \bigcup_{\text{finite}} U_x \supset \bigcup_{g\in G} \partial(gD) \right\rangle = G.$$

It suffices to show $G \subset G'$. Every $h \in G$ is uniquely correspondent to a closed subspace $hD \subset S^n$. If hD and D meet in some good slice U_y, then $h \in G_y$ because an interior point of $U_y \cap hD$ is G_y-conjugate to a (unique) interior point of $U_y \cap D$ and G_y acts effectively on U_y. It is similar when $hD \cap D = \emptyset$, since G is finite and S^n is connected. Therefore, $G \subset G'$. □

References

[Bre72] Glen E Bredon. *Introduction to compact transformation groups*. Academic press, 1972 (↑ 3).

[CGS24] Colin Crowley, Tao Gong, and Connor Simpson. "Toric varieties modulo reflections". In: *arXiv preprint arXiv:2410.14653* (2024) (↑ 1).

[GHL25] Tao Gui, Hongsheng Hu, and Minhua Liu. "Weyl group symmetries of the toric variety associated with Weyl chambers". In: *Bulletin of the London Mathematical Society* (July 2025). DOI: `10.1112/blms.70144` (↑ 1).

[Gon24] Tao Gong. *Homotopy Types Of Toric Orbifolds From Weyl Polytopes*. 2024. arXiv: `2407.16070 [math.AT]` (↑ 1).

[Hor+24] Tatsuya Horiguchi, Mikiya Masuda, John Shareshian, and Jongbaek Song. "Toric orbifolds associated with partitioned weight polytopes in classical types". In: *Selecta Mathematica* 30.5 (2024), p. 84 (↑ 1).

[Hum92] James E Humphreys. *Reflection groups and Coxeter groups*. 29. Cambridge university press, 1992 (↑ 1).

[Son22] Jongbaek Song. "Toric Surfaces with Reflection Symmetries". In: *Proceedings of the Steklov Institute of Mathematics* 318.1 (2022), pp. 161–174 (↑ 1).